Experimental characterization of stochastic resonance in coupled chaotic circuits

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I. Introduction

Stochastic Resonance is a phenomenon observed in nonlinear systems whereby an input signal is optimized due to the beneficial effect of noise. This phenomenon has been observed in many real-world systems[1] including lasers, chemical reactions and biological systems and has many potential applications. This report will focus on the phenomenon of phase synchronization in coupled chaotic systems and will present a metric which gives a better indication of the optimum level of phase synchronization in the presence of noise. This metric, known as the average phase synchronization time will result in a curve that is cusp shaped which gives a narrower range of optimum noise levels than the bell shaped curves obtained for other metrics such as signal-to-noise ratio. Finally experimental evidence for the theory will be presented using coupled Chua circuits along with a discussion of the phenomena observed therein.

II. Stochastic Resonance

Typically noise is considered to be a hindrance to the operation of circuits and systems and most systems are designed to eliminate as much noise as possible. However, noise can also be beneficial in certain systems under the right circumstances due to the phenomenon known as stochastic resonance.

Stochastic resonance can be explained simply using a double-well potential as shown in Fig 1. Consider a heavily damped particle of mass m moving in a symmetric double-well potential where the viscous friction is strong enough to allow us to neglect the acceleration of the particle for a weak driving force. The effect of a weak periodic driving is to tilt the double-well potential back and forth with the result that the potential barriers of the wells are raised and lowered in a complementary manner. This has the effect of lowering the energy barrier of the particle in the shallow well to a level E_b which is less than the original energy required to move into the other well.

If noise is introduced to this system, a sufficiently strong level of noise can cause the particle to overcome the energy barrier E_b and 'hop' into the deeper potential well. At the next half cycle the potential well is tilted again with the result that noise can again push the particle into the adjacent well. Thus a sufficient level of noise will cause the particle motion to synchronize with the periodic driving.

If we consider two coupled systems driving by the same periodic driving, the above argument extended to two double-well potentials still explains the phenomenon of SR with the added caveat that since the

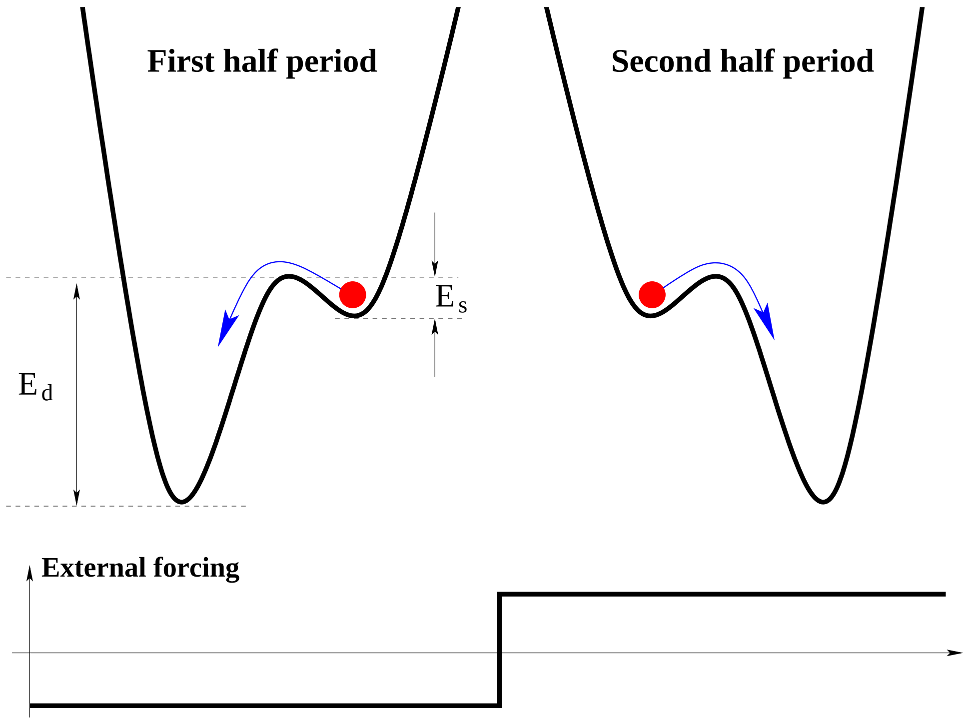

Fig. 1. A tilted double-well potential as a result of external driving. The potential well is tilted every half cycle due to the periodic driving.

particles in each double-well synchronize with the same periodic driving, they will also synchronize with each other at the optimum noise level.

III. Phase synchronization in chaotic oscillators

Phase synchronization is a particular case of stochastic resonance where we consider the synchronization between the phase angles of two systems, or between the input and output of a single system. Obviously this can be considered only for systems that permit a phase angle to be defined.

A. Determination of the phase angle

Consider a chaotic dynamical system defined by a system of N - ordinary differential equations:

$$d\mathbf{x}/\mathbf{dt} = \mathbf{F}(\mathbf{x}) \tag{1}$$

where $\mathbf{x}(\mathbf{t}) = [\mathbf{x^{(1)}}(\mathbf{t}), \mathbf{x^{(2)}}(\mathbf{t}), \ldots, \mathbf{x^{(N)}}(\mathbf{t})]$. The time-series plot for one of the dynamical variables $x^{(i)}(t)$ is shown in Fig a where the amplitudes vary in a chaotic manner. Fig b shows the phase space plot of the outputs of the system where it can be seen that the trajectory is created by a point circling around an orbit in a single direction. Each time an orbit is completed, it can be considered to be a shift in the phase angle by 2π radians. The instantaneous phase can be computed as:

$$\phi(t) = tan^{-1}[y(t) - \langle y \rangle)/(x(t) - \langle x \rangle)] \tag{2}$$

where $\langle y \rangle$ and $\langle x \rangle$ represent time averages of $y(t)$ and $x(t)$. For experimental data, the phase of the signal $x(t)$ can be computed easily by evaluating the analytic signal $x_a(t)$ given by the following relation:

$$x_a(t) = x(t) + jH[x(t)] \tag{3}$$

where $H[x(t)]$ is the Hilbert transform of the signal $x(t)$ given by:

$$H[x(t)] = P.V.[\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{x(t')}{t-t'}dt'] \tag{4}$$

where $P.V$ denotes the Cauchy principal value of the integral. The phase can be computed as the angle of the analytic function. Since the phase is considered to increase by 2π radians every cycle, the phase angle obtained above should be unwrapped to obtain the true phase.

B. Phase synchronization

Now consider two mutually coupled oscillators with instantaneous phases $\phi_1(t)$ and $\phi_2(t)$. For chaotic oscillators we say that the phases of the two oscillators are 'locked' if

$$|\phi_1(t) - \phi_2(t)| \leq 2\pi \tag{5}$$

If the fluctuations $\phi_1(t)$ and $\phi_2(t)$ are chaotic, the difference $\Delta\phi$ will also fluctuate in a chaotic way. The above definition implies that we can consider the phases to be synchronized as long as $\Delta\phi$ is contained within one phase cycle, i.e. 2π.

If the two oscillators operate independently without any synchronization, the phase difference $\Delta\phi = (\omega_1 - \omega_2)t \propto t$ i.e., the phase difference will grow with time. If the oscillators are completely synchronized, $\Delta\phi$ will remain bounded by 2π. However, if the oscillators are not completely synchronized, there will be regions of synchronization where the phase difference does not exceed 2π, but there will be intermittent jumps of $\pm 2\pi$. The time duration of synchronization appears to be random, but it can be expected that the average time duration between such phase jumps will grow as the degree of synchronization (i.e., coupling) grows.

IV. CHARACTERIZATION OF STOCHASTIC RESONANCE

Typical characterizations of stochastic resonance (including phase synchronization) are the Signal-to-noise ratio (SNR), correlation, entropy etc. Since our interest lies in finding the optimum level of noise for SR, these characterizations typically represent the variation of these quantities with noise. If noise can indeed cause one of these quantities to maximize, we can find out the level (or range of levels) which will enhance the response of the system to a weak input signal.

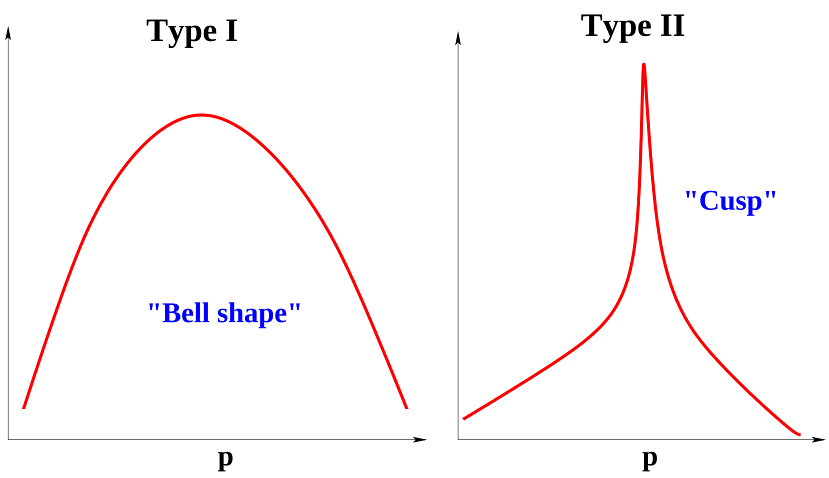

Fig. 2. Sensitivity of a quantity to noise levels with an exhibition of a) bell-shaped behavior and b)cusp-shaped behavior.

If the quantity being used to characterize SR varies smoothly around the optimum noise level, a *bell-shaped* behavior can be observed as seen in Fig 2a. Obviously a bell shaped variation does not give a narrow range of noise levels with optimum sensitivity since the peak is not sharp. If we are interested in applications where the optimum noise level is obtained by an observation of one of the quantities, say SNR, then a small error in measurement of the quantity will lead to a large error in the noise level. Therefore it is desirable to define a quantity that is highly sensitive to the noise level and will exhibit a sharp rise at the optimum value, i.e. a *cusp-shaped* behavior as shown in Fig 2b.

A. Characterization using phase synchronization

Phase synchronization is an important manifestation of the phenomenon of stochastic resonance and has many applications in communication systems. The utility of using phase synchronization to study SR also lies in the fact that it is a more sensitive measure w.r.t. the noise level than traditional methods such as SNR. This also means that if phase synchronization is used as a metric to identify the optimum noise level for a system, it will be more precise than other methods.

It has been shown recently[2] that near the optimal noise level, the average phase synchronization time $\langle\tau\rangle$ exhibits a cusp behavior with a sharp maximum and a steep rolloff. It has also been suggested that this behavior is general to systems exhibiting SR even though the specific levels shown by $\langle\tau\rangle$ depend on the specific system and its parameters. This has been shown to be true numerically for a single oscillator where synchronization is observed between the oscillator output and the periodic driving. We are interested in finding out experimentally if the same synchronization behavior can be observed for a coupled oscillator system driven by a common periodic signal. We shall verify this using coupled chua oscillators operating in non-autonomous modes.

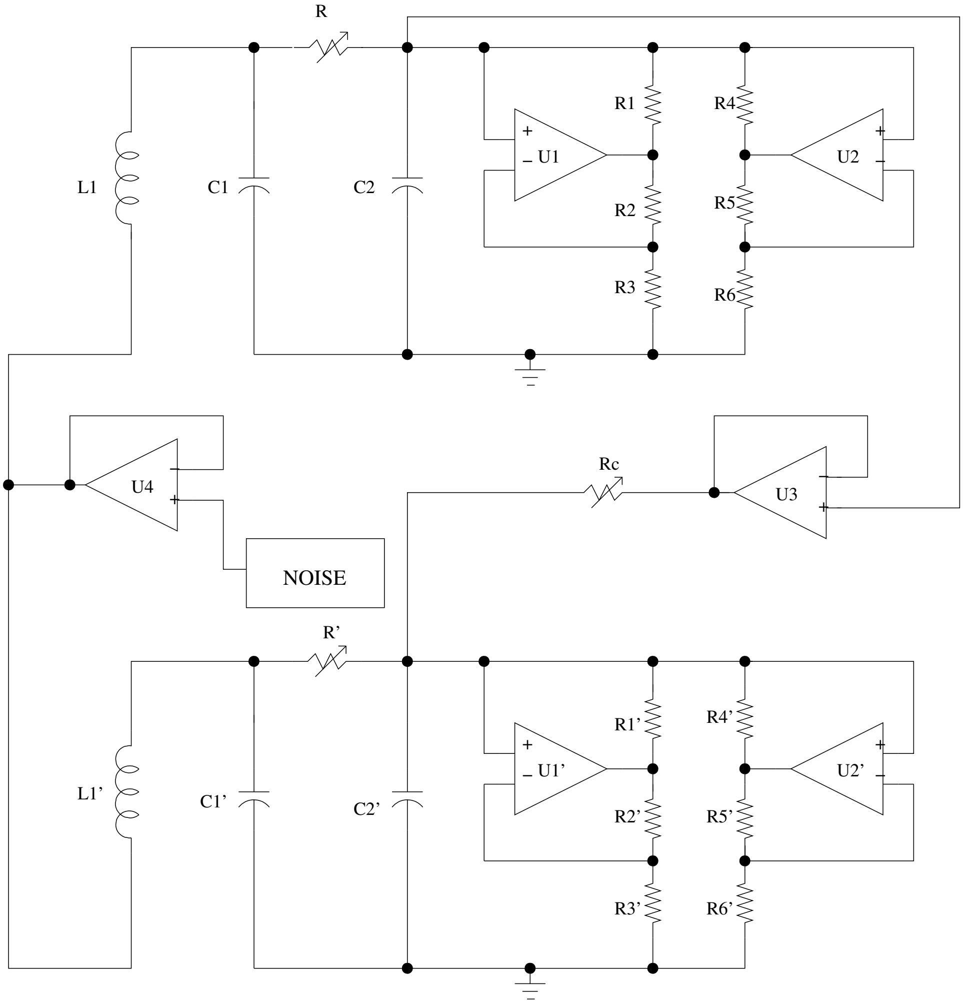

Fig. 3. Circuit diagram of two coupled Chua circuits driven by a common noise source

V. Experimental Verification

A. Coupled Chua Circuits

The experimental setup consists of two identical Chua generators coupled in a unidirectional way and driven by a common noise source as shown in Fig.3. The inductor L1, capacitors C1 and C2, the potentiometer R and the Chua diode realized using two operational amplifiers U1 and U2 along with the resistors R1-R6, form a single Chua circuit. An identical Chua circuit is built using the same component values. The two Chua circuits are unidirectionally coupled using the operational amplifier U3 and the potentiometer R_c. The value of R_c could be adjusted to vary the coupling strength between the two circuits. A common noise is added to the inductors L1 and L1' of the two circuits respectively using an opamp U4 as a buffer. The addition of noise to the Chua circuit makes it non-autonomous. The equations of the complete circuit are given by:

$$
\begin{aligned}
L_1 \frac{di_{L_1}}{dt} &= -v_{C_1} + \xi(t) \\
C_1 \frac{dv_{C_1}}{dt} &= i_{L_1} - \frac{(v_{C_1} - v_{C_2})}{R} \\
C_2 \frac{dv_{C_2}}{dt} &= \frac{(v_{C_1} - v_{C_2})}{R} - f(v_{C_2}) \\
{L_1}' \frac{d{i_{L_1}}'}{dt} &= -v'_{C_1} + \xi(t) \\
{C_1}' \frac{d{v_{C_1}}'}{dt} &= {i_{L_1}}' - \frac{({v_{C_1}}' - {v_{C_2}}')}{R'} \\
{C_2}' \frac{d{v_{C_2}}'}{dt} &= \frac{({v_{C_1}}' - {v_{C_2}}')}{R'} - f'({v_{C_2}}') - \frac{(v_{C_2} - {v_{C_2}}')}{R_c}
\end{aligned}
\tag{6}
$$

where $\xi(t)$ represents the noise added to the circuit while $f(.)$ and $f'(.)$, the equations of the Chua diodes of the two circuits respectively.

$$
\begin{aligned}
f(x) &= m_0 x + \frac{(m_1 - m_0)(|x + B_p| - |x - B_p|)}{2} \\
f'(x) &= m'_0 x + \frac{(m'_1 - m'_0)(|x + B_p| - |x - B_p|)}{2}
\end{aligned}
$$

$$
m_0 = (-\frac{R_2}{R_1 R_3} - \frac{R_5}{R_4 R_6})R \ , \ m_1 = (-\frac{1}{R_4} - \frac{R_2}{R_1 R_3})R \tag{7}
$$

$$
m'_0 = (-\frac{R'_2}{R'_1 R'_3} - \frac{R'_5}{R'_4 R'_6})R' \ , \ m'_1 = (-\frac{1}{R'_4} - \frac{R'_2}{R'_1 R'_3})R' \tag{8}
$$

The Chua diodes of both the circuits were realized using a single TL084 chip and the resistors and buffers U3 and U4, using TL082. The component values were fixed as L1=18mH, C1=100nF, C2=10nF, R1=R2=22KΩ, R3=3.3KΩ, R4=R5=220Ω and R6=2.2KΩ. The corresponding parameters of the second circuit were fixed to be the same.The potentiometer R was varied to observe different behaviors (periodic and chaotic).

Initially, the coupling was removed and no noise signal was given. In this case, the circuit behaved as two independent autonomous generators. The potentiometers R and R' were fixed such that the phase-space trajectory of v_{C_1} and v_{C_2} of each circuit was a double scroll. The onset of the double scroll occurs at slightly different values of R and R' although all the other parameters of the circuits were made identical. This can be attributed to the parametric variations of the components introduced during their manufacture and the sensitivity of chaotic circuits to those variations. The value of R was fixed as 1.763KΩ and that of R' as 1.736KΩ such that both circuits were showing a double-scroll attractor initially.

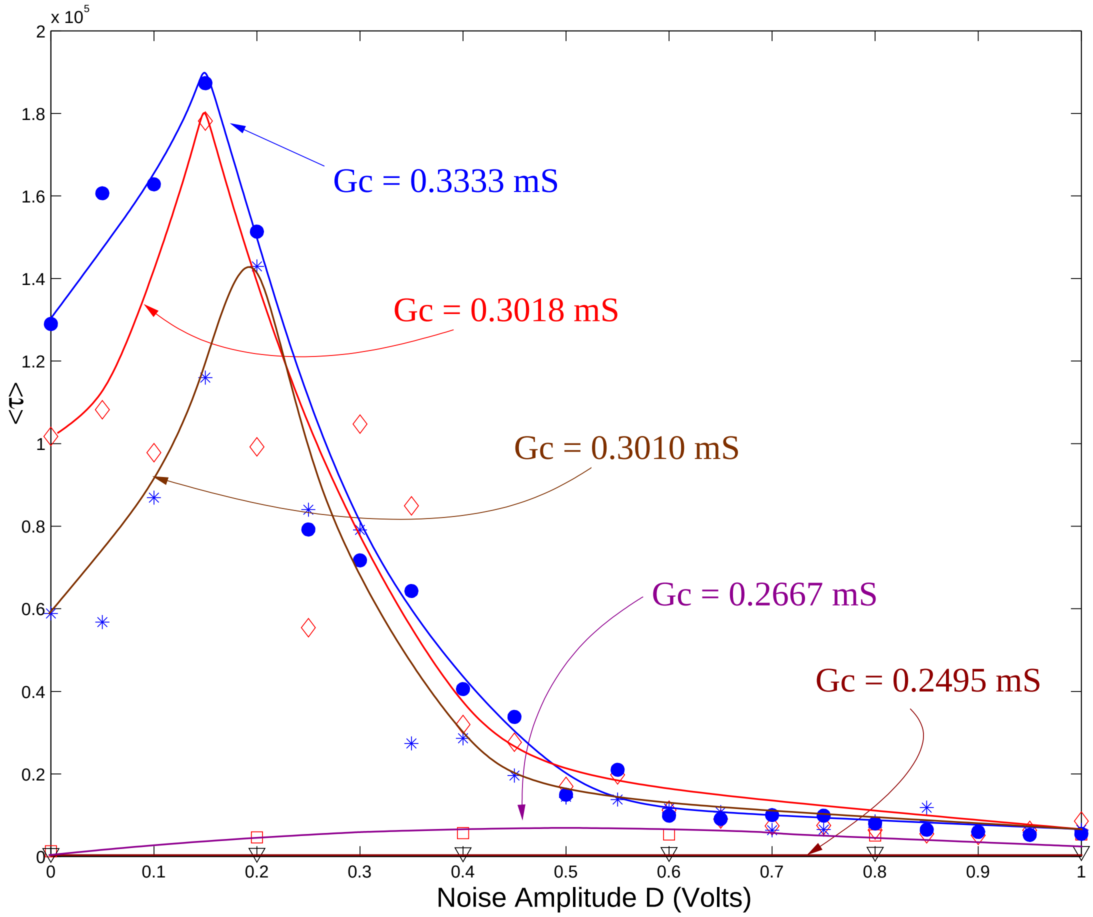

Fig. 4. Plot of the average synchronization time $\langle \tau \rangle$ for various coupling strengths. For very low coupling strengths the curve is almost flat and it narrows as the coupling strength is increased. Also, the peak of the curve shifts toward the left as the coupling strength is increased since a lower noise level can optimize the system response.

The circuits were coupled and the coupling strength was fixed by the value of the potentiometer R_c. The coupling strength ϵ_c is directly proportional to the coupling conductance G_c where $G_c = \frac{1}{R_c}$. The noise voltage was increased from 0 to $1V(p-p)$ and the voltages v_{C_2} and $v_{C_2'}$ were acquired using the National Instrument DAQ system. The post processing of the measured signals were done using MATLAB to calculate the average synchronization time.

VI. Results

The behavior of the average synchronization time with the noise voltage was studied over different values of coupling strengths. For low value of coupling strengths ($G_c = 0.2495 \; and \; 0.2667 \; mS$), it turns out to be almost a flat response for the noise range of 0 - 1 $V_p - p$. However, when we increase the noise intensity to higher values, it shows a bell-shaped dependence on it in accordance with the regular stochastic resonant behavior. So, it is seen that noise enhances the phase synchronization of weakly coupled oscillators in a very smooth way.

However, as we increase the coupling strength, the average synchronization time shows a cusp-shaped

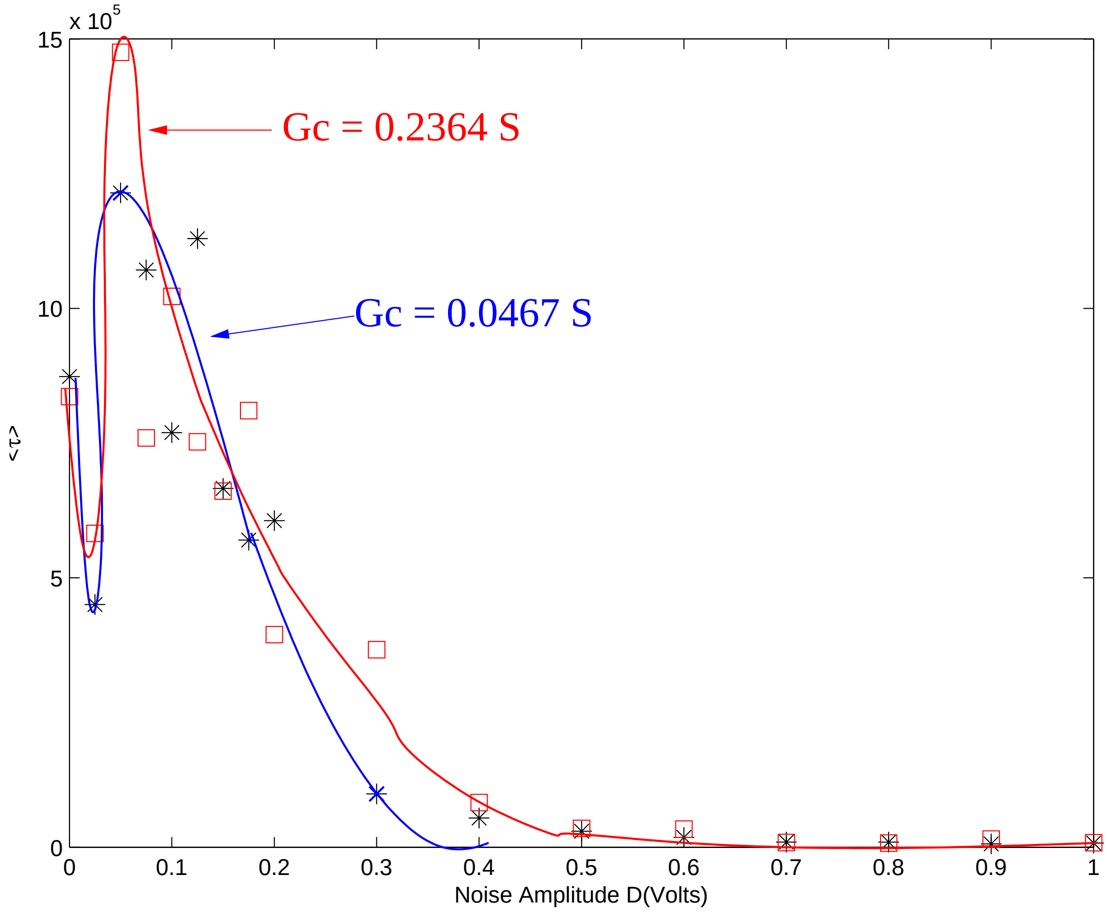

Fig. 5. Plots of the average synchronization time $\langle\tau\rangle$ for very high coupling strengths. $\langle\tau\rangle$ increases to very high levels but it also displays an initial 'dip'.

dependence on the noise intensity as could be seen in Fig 4 for G_c values of 0.3333,0.3018 and 0.3010 mS respectively. Moreover, as we increase the coupling strengths between the two circuits, we could observe the following behaviors: 1) The average synchronization time at zero noise level increases. This is due to the increase in the synchronization between the two circuits with the increase in the coupling strengths 2) The peak average synchronization time at the optimal noise intensity (the point at which resonance occurs) increases. This is because of the cusp-shaped dependence of it to the noise level and that it starts from a higher value as was discussed before and 3) The noise intensity at which resonance occurs is decreased. Considering all the three behaviors discussed above, we could see the cusp-shaped behavior shifting to the left towards the zero noise level and with higher peaks as we increase the coupling strengths as could be seen in the family of curves as shown in Fig 4. As we go to very high coupling strengths, for G_c values of 0.0467 and 0.2364 S, the synchronization shows a highly sensitive dependence to the noise intensity as could be seen in Fig 5. We could also make the following observations: 1) The optimal noise level at which the peak occurs becomes very close to zero 2) Moreover, we could also see a dip in the synchronization time and then an increase in the same with the increase in the noise intensity.

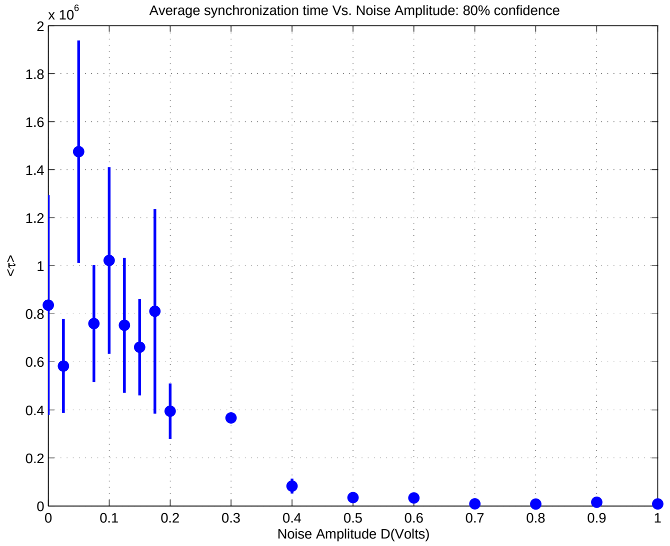

Fig. 6. Plot of the average synchronization time $\langle\tau\rangle$ with the error bars for 80% confidence level shown. The confidence levels are estimated as $\bar{x} \pm 1.282\sigma$ where $\bar{x}$ and σ represent the mean and standard deviation of the data respectively.

This behavior is not seen in the weakly coupled and slightly higher coupled cases(Fig 4). It could also be seen that the optimal noise level at which resonance occurs remains to be the same for different values of coupling strengths as it becomes difficult to resolve those differences experimentally.

The error bars of the plot of $\langle\tau\rangle$ are shown in Fig. 6 for an estimated 80% confidence level. The confidence level was estimated as $\bar{x} \pm 1.282\sigma$ where $\bar{x}$ and σ represent the mean and standard deviation of the data respectively. It can be seen that the error bars are very small for high values of noise since the low phase synchronization causes more frequent phase slips and hence more values of $\langle\tau\rangle$. Around the optimum noise level, the high degree of phase synchronization gives us very few intervals due to which the estimation of $\langle\tau\rangle$ has a larger standard deviation σ.

VII. Summary and Conclusions

In summary, it is seen that the average synchronization time is a good measure to characterize the stochastic resonant behavior of coupled chaotic circuits. It could also be concluded that noise enhances the phase synchronization of weakly coupled chaotic circuits in a very smooth way as was seen from the bell-shaped dependence of the average synchronization time with the noise intensity and the optimal noise intensity at which the resonance occurs is higher when compared to the strongly coupled cases. On the contrary, the phase synchronization shows a highly sensitive dependence on the noise intensity as was seen in the cusp-shaped dependence of the average synchronization time with noise intensity. And, the optimal noise level at which resonance occurs decreases with the increase in the strength of the coupling between the two circuits. At very high coupling strengths, an interesting dependence - an initial dip and then a peaking of the average synchronization time with the increase in the noise intensity was observed.

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