INDUCING CHAOS IN SYSTEMS USING RESONANT PERTURBATIONS

by

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December 2004

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ABSTRACT

A chaotic system is one whose long term behavior is unpredictable and is sensitive to the initial conditions. This thesis explores the idea of inducing chaos in systems as a means to render them inoperable. The idea is to use resonant external perturbations to drive non-linear oscillators into progressively higher resonant states and eventually into chaos. The advantage of this approach over conventional schemes which use high amplitude signals to jam circuits is that relatively weaker signals can be used to induce chaos in systems that need to operate in a restricted parameter regime of order. Weak periodic signals with time varying frequency are used to induce chaos in circuits which are by themselves incapable of exhibiting any chaotic behavior in the region of operation.

This thesis also presents an experimental verification of the idea of inducing chaos using resonant perturbations by using a Duffing oscillator along with a phase locked loop. It is seen that resonant perturbations can indeed drive the system into chaos from a periodic state while random perturbations cannot achieve the same effect. This setup makes it possible to design systems that are small and consume very little power which makes them suitable as defensive systems on mobile platforms such as aircraft.
To Daddy and Mummy
ACKNOWLEDGMENTS

I thank my advisor Dr. Ying-Cheng Lai for giving me this opportunity to work with him on this very interesting topic. I also thank Dr. Cihan Tepedelenlioglu and Dr. Thomas Taylor for generously agreeing to serve on my thesis committee. Thanks to Satish Krishnamoorthy for his help with the circuits and for very fruitful discussions. Thanks also to Liqiang Zhu and Younghae Do for many interesting discussions which helped make this project a success. I would also like to thank Christopher Silva and Albert Young for providing the design of the Duffing oscillator.

It has been a great experience working with all my colleagues in the nonlinear dynamics group at ASU and thanks are due to them for being supportive and making the whole process of research so much fun. Finally, I would like to thank all my friends and roommates for making it all worthwhile.
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1. What is Chaos?

Chaos is a term used to describe behavior that is seemingly random, but has an underlying mathematical order to it. Chaos is very common in nature, but is often mistaken for random behavior. Chaos can occur only in nonlinear systems and is characterized by a breakdown of predictability known as sensitive dependence on initial conditions which is the most important distinguishing feature of chaos. This implies that even though chaotic systems are deterministic (unlike systems exhibiting random behavior), even the smallest difference in initial state can cause a huge difference in the end state. This is famously described as the butterfly effect whereby the flapping of a butterfly’s wings in one part of the world can eventually lead to a radical change in weather in another part. Long term predictability of chaotic systems is impossible since all numerical calculations have a finite non-zero error which will diverge over time and make the predictions unreliable. The following properties are shared by all chaotic systems:
1. Chaos can occur only in deterministic nonlinear dynamical systems.

2. Chaotic behavior looks complicated and irregular but has an infinite number of unstable periodic patterns embedded in it.

3. Chaotic behavior is sensitive to initial conditions.

2. Maps and Dynamical Systems

Even though long term predictability is not usually achievable in chaotic systems, they are deterministic (and not probabilistic like stochastic systems) which means that future states of the system can be uniquely determined from past states. If the evolution of a system in time can be described by a rule, it is called a dynamical system. If time is a continuous variable, the system can be described by a set of differential equations and it is called a continuous-time dynamical system. The usual way to describe such a system is to use a set of first-order autonomous ordinary differential equations such as:

\[
\begin{align*}
\dot{x}_1 &= F_1(x_1, x_2, \ldots, x_N), \\
\dot{x}_2 &= F_2(x_1, x_2, \ldots, x_N), \\
&\vdots \\
\dot{x}_N &= F_N(x_1, x_2, \ldots, x_N)
\end{align*}
\] (1.1)

where $F_1, F_2, \ldots, F_N$ represent functions of the dynamical variables $x_1, x_2, \ldots, x_N$.

In principle, given any initial condition, such a system can be evolved forward in time. The $N$-dimensional space occupied by the system as it evolves in time is known as the phase-space and the path followed by the system is known as the orbit. It has been shown that for such a system to exhibit chaos, the necessary condition is that $N \geq 3$. If
the rule that evolves the system forward in time is applied to discrete time steps, a \textit{map} is obtained. A map is represented as:

\[ x_{n+1} = M(x_n) \]  

(1.2)

where \( x_n \) is an \( N \)-dimensional vector and \( M \) is the map that defines the system. Thus, a dynamical system can be represented either by a set of ODEs or by a map. The nonlinear systems described in our study will be represented by a set of ODEs as shown in Eqn. 1.1.

3. Characterizing Chaos

As mentioned earlier, an important hallmark of chaos is its sensitive dependence on initial conditions. For a periodic attractor, two nearby points will maintain the distance between them with the progression of time, while two nearby points on a chaotic attractor will diverge continuously with time i.e., the distance between them will increase monotonically. This is a useful characteristic which helps distinguish chaos from periodic motion or even purely random motion. For purely random motion, two nearby points will not diverge monotonically, and instead will fluctuate randomly.

3.1. Lyapunov Exponents. The Lyapunov exponent of a one-dimensional map is the average exponential rate of divergence of infinitesimally nearby initial conditions [1]. In general, Lyapunov exponents are a measure of the phase-space volume expansion along a trajectory. Consider two initial points \( x_0 \) and \( x_0 + dx_0 \) in a one-dimensional map which evolve with time. If the separation between the points is \( dx_T \) after the map has been iterated \( T \) times, the Lyapunov exponent \( h \) can be defined as:
System with unknown state space

Figure 1. Time series analysis for a system using a single measured quantity. The measured time series is embedded to form a reconstructed phase space which can be analyzed using time-series algorithms to find the Lyapunov exponents.

\[ h = \lim_{T \to \infty} \frac{1}{T} \ln \left| \frac{dx_T}{dx_0} \right| \]  

(1.3)

It can be seen from Eq. 1.3 that if \( h > 0 \), two nearby points separated by a distance \( \delta_0 \) will diverge from each other after \( n \) iterations as \( \delta_n \approx \delta_0 e^{hn} \) if \( n \) is sufficiently large. Thus a positive Lyapunov exponent is a good indicator of chaos.

In general, there is one Lyapunov exponent per dimension for a dynamical system. For an \( N \)-dimensional system, the \( N \) Lyapunov exponents are typically ordered from largest to smallest as \( \lambda_1 > \lambda_2 > \ldots > \lambda_N \). It is therefore generally sufficient to check if the maximum Lyapunov exponent \( \lambda_1 \) is positive or not to verify that the system output is indeed chaotic.
3.2. **Time-Series Analysis.** If the system dynamics are known, analytical methods can be used to compute the Lyapunov exponents using the system equations. The number of Lyapunov exponents, which is equal to the system dimension is given by the number of variables in the dynamical model. However, while dealing with experiments, the state space is not normally known. For example, if we treat the nonlinear system as a ‘black-box’, we may be able to measure only one or two variables as a function of time. To perform any sort of analysis on this data, it is necessary to first reconstruct the original system using the available data.

3.2.1. *Embedding.* It has been shown [2, 3] that it is indeed possible to arrive at an attractor that can characterize the original system, using only a single measured component of the system. The preferred method for reconstructing the phase space is the *delay coordinates* method which maps every point \( x \) in the original \( N \)-dimensional state space to a point \( y \) in a \( d \)-dimensional state space where \( d \) denotes the *embedding dimension*. It should be noted that the process of *embedding* does not give us back the original state space; instead, we obtain a new state space which has been proved to be equivalent to the original phase space. Since every point in the original state space corresponds to a unique point in the reconstructed space, it is possible to study the evolution of the original system by observing the evolution of the reconstructed system. For this embedding to be reliable, the required embedding dimension \( d \) is related to the original state space dimension \( N' \) of the *attractor* by the relation

\[
d \geq 2N' + 1
\]  

(1.4)

where \( N' \) is the box-counting dimension of the attractor which might be of a lower dimension than the system itself. The choice of embedding dimension is very important
since too small a dimension will not produce reliable results, and too large a dimension will increase the amount of computation required.

3.2.2. Calculation of maximal Lyapunov exponent from embedded time-series. Once a state space has been reconstructed from the measured time-series, it is possible to use various methods to calculate the Lyapunov exponents of the system. These methods can typically be classified [4] as: (1) Jacobian based methods [5, 6, 7, 8, 9] and (2) Direct methods [10, 11, 12, 13, 14]. Direct methods are easy to use and estimate the maximum Lyapunov exponent by directly measuring the divergence of nearby orbits. A simple and robust algorithm which can estimate the largest Lyapunov exponent is the Wolf-algorithm [10] which considers the divergence of an orbit \( x^n \) from a neighboring reference orbit \( y^n \).

The Lyapunov exponent \( \lambda_1 \) can be determined from the growth of the magnitude of the difference vector \( u^k = x^{n+k} - y^{n+k} \) with time. A similar algorithm [11, 12] will be used in the analysis of our data. This algorithm calculates the prediction error as:

\[
p(k) = \frac{1}{N t_s} \sum_{n=1}^{N} \log_2 \left( \frac{\|y^{n+k} - y^{n+n+k}\|}{\|y^n - y^{nn}\|} \right) \quad (1.5)
\]

The largest Lyapunov exponent can be estimated from the plot of \( p(k) \) as the slope in the appropriate region. (The initial region corresponds to the transient region where the difference vector \( u^k \) does not yet point in the direction corresponding to the largest Lyapunov exponent). This can be understood very simply if we consider the fact that if the system is periodic, the prediction error should not increase with time, while for a chaotic system where the neighboring orbits diverge, the prediction error should increase with time. Thus the slope of the prediction error for a periodic case would ideally be zero, while the slope for a chaotic case would be positive. For experimental data, the Lyapunov exponent is calculated using this algorithm for many ensembles of data to give a statistical indication.
of whether $\lambda_1$ is positive or $\approx 0$. The procedure for finding the Lyapunov exponent from a time-series is summarized in Fig. 1.

4. Chaos and the real world

Chaotic systems can be described mathematically and they have an infinite number of unstable periodic patterns hidden in them, but chaos still presents itself as irregular dynamics and is therefore considered to be an anomaly in the normal operation of systems. However this is not completely true and many systems such as the brain operate in a chaotic manner and ordered operation may be anomalous. Chaos is quite common in biological systems such as cardiac systems [15] and is generally required for the normal operation of many such systems.

It has been shown that the evolution of the solar system is actually chaotic [16]. In fact the origin of chaos theory lies in the study conducted into the three body problem by Henri Poincaré in 1890. The study of the mathematical basis of chaos theory has developed to the point where there are innovative applications in many different fields [17]. Chaos control has been successfully used in lasers, control of chemical reactions and cardiac arrhythmia. Chaotic synchronization, where two artificially generated chaotic sequences are synchronized, can be used for message encoding and decoding similar to spread-spectrum communication systems using synchronized chaotic sequences. These chaotic sequences can also be used for message encryption. Chaos also has applications in short term prediction of systems that can be reliably identified as chaotic systems such as the weather or the stock markets.

The application of chaos presented in this thesis will focus on using the disruptive nature of chaos to our advantage as a defensive mechanism. Chaos will be used to alter
the normal functioning of circuits to the point where they are rendered inoperable using only tiny perturbations. A theoretical treatment of the theory of inducing chaos using resonant perturbations along with experimental verification of this idea will be presented in the succeeding chapters.
CHAPTER 2

Inducing Chaos

"It turns out that an eerie type of chaos can lurk just behind a facade of order - and yet, deep inside the chaos lurks an even eerier type of order”

-Douglas Hostadter

1. Problem statement

Even though chaotic behavior is usually a hindrance to the normal operation of systems, it can be put to good use as a defensive mechanism. Most modern weaponry incorporates sophisticated electronic devices that are crucial for reliable operation. These devices use complex circuits that can operate well only in restricted parameter regimes that can be classified as stable or regular. Any complicated and seemingly random behavior in these systems will likely create enough confusion to prevent any reliable operation such as target acquisition or tracking.

Since the circuitry is not readily accessible, it will not be possible to change any circuit parameters to hinder their regular operation. It may be possible to confuse the electronics by irradiating it with electromagnetic waves of large amplitude, thereby jamming the circuitry. A more elegant and efficient solution is to use small amplitude perturbations to
induce chaos in the systems thereby achieving the goal of preventing any reliable operation of dependent systems. This is particularly useful since the amplitude of excitations required may be small enough to be incorporated into defense mechanisms on mobile systems, such as combat aircraft.

An electronic circuit can be viewed as a dynamical system as defined in Sec. 2 since their operation is to evolve state variables such as currents/voltages continuously with time. The mathematical concepts of inducing chaos in dynamical systems can thus be readily applied to electronic circuits. We will generate a methodology to induce chaos in a general dynamical system using resonant perturbations and eventually design an experimental setup using an electronic circuit to verify this idea.

2. Principle of using external excitations

Consider a general dynamical system consisting of $N$ autonomous differential equations:

$$\frac{dx}{dt} = f(x)$$  \hspace{1cm} (2.1)

where $x \in \mathbb{R}^N$ is an $N$-dimensional vector with components $x_1, x_2, \ldots, x_N$) and the velocity field $f(x)$ is a nonlinear function. The state equations describing most electronic circuits can in fact be written in the above form. Now imagine that an external excitation is applied so that Eq. 2.1 becomes:

$$\frac{dx}{dt} = F[x, g(t)],$$  \hspace{1cm} (2.2)
where now the velocity field contains an explicit time dependence due to the time-varying excitation \( g(t) \) and, hence, it is nonautonomous. Equation (2.2), however, can be converted into an autonomous system simply by introducing a new \((N + 1)\)-dimensional variable: \( y(t) = [x(t), g(t)] \), where \( y_{N+1} = g(t) \). Thus, the dynamical equation for \( y \) can be written as:

\[
\frac{dy_i}{dt} = F_i(y_1, y_2, \ldots, y_N, y_{N+1}), \quad i = 1, \ldots, N \tag{2.3}
\]

\[
\frac{dy_{N+1}}{dt} = \frac{dg}{dt}.
\]

It can be seen that the system governing the motion of the new variable \( y \) has a phase-space dimension that is one higher than that of the original equation in \( x \) in the absence of any excitations. An increase in the phase-space dimension indicates a possible increase in the complexity of the system and, hence, chaos is more likely to occur in the new system. Thus, in principle, it is thus possible to induce chaos in a nonchaotic system by using external time-dependent excitations. We build upon this principle to generate our methodology for inducing chaos.

### 3. Previous work

There are existing works on inducing or maintaining chaos in nonlinear systems. These can be categorized into two classes: (1) inducing chaos by random noise [18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28] and (2) converting transient chaos into sustained chaos by small perturbations - the problem of maintaining chaos [29, 30, 31, 32, 33, 34].

#### 3.1. Inducing chaos by random noise

In the first class, the main question concerns how chaos can arise under the influence of random noise. The pioneering work
of Crutchfield et al. [18, 19] established that, in the common route to chaos via a cascade of period-doubling bifurcations, noise tends to smooth out the transition and induce chaos in parameter regimes where there is no chaos otherwise. The observability and scaling of fractal structures near the transition to chaos in random maps was addressed in Refs. [23, 24]. Features of transition to chaos in noisy dynamical systems, such as intermittency and a smooth behavior of the Lyapunov exponents, were also found in the transition from strange nonchaotic to strange chaotic attractors in quasiperiodically driven systems [35] and in the bifurcation to chaos with multiple positive Lyapunov exponents in high-dimensional systems [36, 37, 38]. More recently, the mechanism for transition to chaos in continuous-time dynamical systems has been investigated [28] where it is found that nonhyperbolicity plays a fundamental role in shaping the transition. The second class of problems deals with systems in parameter regimes where there are nonattracting chaotic sets that physically lead to transient chaos. That is, under its own evolution, from a random initial condition the system behaves chaotically only for a finite amount time before settling into a nonchaotic attractor.

3.2. Maintaining chaos. Since our goal is to obtain sustained chaos, it is desirable that the system remain in chaos after we take it into a chaotic regime from a stable or periodic one. Chaos can be maintained in the system by applying small perturbations to an available parameter or dynamical variable of the system [29, 33, 34].

The problem of inducing chaos and maintaining it has indeed been studied before, but in all the situations considered, chaos can be generated and/or maintained by using small random perturbations or feedback control, but this can be achieved only when the system is already in a chaotic regime in the sense that there are nonattracting chaotic
sets coexisting with some nonchaotic attractors such as a stable periodic orbit \[28\]. In the type of applications that are addressed here, the problem is very different in that the system of interest is usually \textit{far away from any chaotic regime}. An additional restriction is that the internal parameters or dynamical variables of the system are inaccessible, i.e. the explicit system equations are \textit{unknown}. Under the circumstance the only viable approach to disturb the system is to apply external excitations. It will be demonstrated that this is indeed possible by using the principle of resonance, which is expected to occur generally in nonlinear oscillators.

4. Nonlinear resonance in nondissipative systems

To gain insight, we imagine a simple, linear, conservative (Hamiltonian) system: a harmonic oscillator that is a textbook example in classical mechanics. In this system, the natural oscillating frequency \(\omega\) does not depend on the energy. As a consequence, we can induce an arbitrarily large disturbance in this system by applying an arbitrarily small perturbing force at the fixed resonant frequency \(\omega\). It is important to note that the resonant frequency is the same for all energies, because the natural frequency is constant. This convenient feature is, however, restricted to linear systems. For nonlinear systems, the system’s natural frequency does depend on the energy, and using a perturbing field with a fixed frequency in general cannot generate resonance. Our basic idea is then to change the frequency of the perturbing field so as to “follow” the natural frequency of the system as the energy changes due to perturbation.

We consider here a nondissipative oscillating system with one degree of freedom. Without any external perturbation, the system exhibits simple stable motion; it fundamentally prohibits any chaotic motion. Thus, showing that chaos can be induced in such a
system by small resonant perturbations demonstrates the power of our method.

4.1. Hamiltonian System with fixed-frequency perturbations. The dynamics of the unperturbed Hamiltonian system is described by:

\[
\frac{d^2x}{dt^2} = -\frac{dV(x)}{dx},
\]

where \(x\) is the dynamical variable (e.g. the current or voltage in an electronic circuit), and \(V(x)\) is the potential function. For a harmonic oscillator, \(V(x)\) is a quadratic function. In general, \(V(x)\) can be any differentiable function. We assume, however, that \(V\) has a minimum and a maximum (this is not satisfied by the harmonic potential). Although our method works for any potential satisfying these constraints, in this Report we will focus on the pendulum potential, given by:

\[
V(x) = -\cos(x).
\]

The maxima at \(x = \pm \pi\) define hyperbolic orbits at energy \(E = 1\). The hyperbolic orbits separate regions of confined and non-confined motion. Widespread chaos arises in the vicinity of the hyperbolic orbits, for arbitrarily small perturbations \([39, 40]\). The oscillating frequency of the unperturbed system is a function of the energy: \(\omega = \omega(E)\), where \(\omega\) is defined within the region of confined motion, \(-1 \leq E \leq 1\). The frequency at the minimum is \(\omega(-1) = 1\), and it decreases towards 0 as \(E\) approaches 1, because it takes an infinite amount of time for the hyperbolic orbits to go from one maximum to the other, say from \(x = +\pi\) to \(x = -\pi\). We stress that this last feature is not particular of the pendulum potential, but it is true of all hyperbolic orbits: they have an infinitely long period. As the energy at which there is a hyperbolic orbit is approached, the period diverges, and the frequency goes to zero. This is important for our method.
Consider now that the system is set up with an initial energy $E_0 < 1$. In the absence of perturbations, it will keep oscillating with this constant energy. Our goal is to apply a small perturbation so that the energy is increased toward $E_{\text{max}}$, where the homoclinic orbits lie around which there is sustained chaos. If we just apply a perturbation with a fixed frequency equal to the initial natural frequency $\nu_0 = \omega(E_0)$, the system will rapidly fall out of resonance, as we explained above. The result is that the energy $E$ oscillates around $E_0$, with an amplitude that decreases with the strength of the perturbation. This is shown in Fig. 2, where we use Eq. (2.4) with an added constant sinusoidal term $F \sin(\nu_0 t + \phi_0)$, and the energy is plotted as a function of the number of oscillations of the external excitation. Since we are interested in weak perturbations (small $F$), it is clear that we will not be able to reach $E_{\text{max}}$ in this way.

4.2. Hamiltonian system with varying-frequency perturbation. The key observation is that the system’s natural frequency changes with the energy. We must therefore change the frequency of the perturbation so that it always matches the natural frequency, thus ensuring that the resonant condition be satisfied at all times. The frequency of the external excitation thus changes with time, and we write $\nu(t)$. The form of $\nu(t)$ cannot be written down explicitly, because it is adjusted in response to the time variation of the natural frequency of the system. The equations of motion of the perturbed system can be written as:

$$\frac{d^2 x}{dt^2} = -\frac{dV(x)}{dx} + F \sin[\nu(t)t + \phi(t)],$$

where $\phi(t)$ is a time-dependent phase that will be discussed shortly.

4.2.1. Determination of resonant frequency. Although for the particular case of the potential (2.5), the frequency as a function of the energy is a known analytical function
Figure 2. Energy of the oscillator as a function of number of oscillation period for an excitation at constant frequency, at resonance with the initial frequency. The initial conditions are \( x(0) = 0.5, \ x'(0) = 0 \), with a corresponding energy of \( E \approx -0.88 \), with \( F = 10^{-3} \).
(expressed in terms of the elliptical functions) [40], we want to keep our method as generic as possible, and so we assume such dependency is not known. In fact, we do not assume any knowledge of the potential, other than the fact that it has a minimum (with a region of confined, oscillating motion) and a hyperbolic orbit. In other words, we only require that the underlying system be oscillatory. Therefore the only way to determine the natural frequency of the system at a given time is through the observed dynamics, for instance, through the observation of the dynamical variable \( x(t) \). We cannot measure the period directly from the dynamics, since the forcing term in Eq. (2.6) makes the motion aperiodic. However, since the perturbation is small \( F \ll 1 \), at any given time the motion is almost periodic, meaning that the energy changes only very slowly with time. Typically, the system oscillates many times with only a small change in \( E \) and, hence, the resonant frequency changes very little as well. Using this fact, we define \( \nu \) for a given time \( t \) as the average over the past \( \Delta n \) oscillations, where \( \Delta n \) is small enough so that the energy does not change appreciably in the corresponding time interval. In this way \( \nu \) is defined purely in terms of the observed quantities of the system, and any previous knowledge of the potential and/or the equations of motion are not assumed. The only requirement is that the average oscillating frequency of the system as a function of time be measured. In principle, the forcing term in Eq. (2.6) involves a time delay because the forcing is equivalent to a memory term. In our simulations, we consider only one oscillation in the past. The results we present seem to be independent of \( \Delta n \), to within the constraint mentioned.

4.2.2. Need for phase synchronization. Tuning the forcing frequency is not sufficient to induce chaos. We also need to control the phase. Since we want the energy to increase in time so that the system approaches the hyperbolic orbit, we have to adjust the phase \( \phi \) so that the forcing term is always in phase with the system’s oscillation. We do that by
making adjustments in discrete times: every time \( x \) crosses 0 in the positive direction, we change \( \phi \) in Eq. (2.6) so that the forcing term is in phase with \( x(t) \). For a real circuit, this could be achieved continuously by a phase-locking scheme. However, we note that the adjustment at each oscillation is small, because the oscillations depart only a little from periodic motion. Therefore, we believe the results would be essentially the same for both approaches. Imposing this phase-adjusting mechanism, we ensure that energy is always transferred from the perturbing force to the system, and not the other way around. If we do not do that, the phase would drift in time, and the energy would not increase monotonically in time as we wish, but would instead oscillate more or less randomly.

**4.3. Numerical results.** Now we apply the method described above to the system given by Eq. (2.6), with the potential \( V \) given by Eq. (2.5). The system is initialized with the same initial conditions as in Fig. 2. The plot of energy versus time is shown in Fig. 3(a). We see that the method does indeed work: the energy increases monotonically with time, as the perturbation injects energy into the system, until it reaches the threshold \( E = E_{\text{max}} \), at which chaos occurs. Figure 3(b) shows the frequency variation corresponding to the changing energy, where it decreases when the energy increases and remains constant when the energy reaches unity. We find that \( E \) will eventually reach \( E_{\text{max}} \) for any values of \( F \), no matter how small (this is not true for dissipative systems, as we shall describe below). For perturbations of different magnitude, the difference is that the smaller the value of \( F \), the longer it takes for \( E \) to reach \( E_{\text{max}} \). For small enough \( F \), we find that, approximately, this time is inversely proportional to \( F \). Let \( n_{\text{max}} \) be the number of oscillations of the external excitations required for energy \( E \) to reach unity. Figure 4 shows the relationship between
Figure 3. (a) Energy of the oscillator as a function of oscillation period for $F = 10^{-4}$. The phase $\phi$ in Eq. (2.6) is chosen as in phase with $x(t)$ for $E < 1.0$ and it remains constant for $E \approx 1.0$. Since Eq. (2.6) represents a Hamiltonian system, the energy will oscillate about $E = 1$ under the influence of external perturbation $F \sin(\nu t + \phi)$. (b) Frequency of the applied perturbation as a function of oscillation period.
$n_{max}$ and the perturbation strength $F$, where we observe the following:

$$n_{max} \sim F^{-1}. \quad (2.7)$$

This relation can be understood by writing down the energy function of the system: $E(t) = u^2(t)/2 + V(x)$, where $u(t) = dx/dt$ is the velocity of the particle. Taking time derivative of $E(t)$, in combination with Eq. (2.6), yields:

$$\frac{dE}{dt} = u(t)\{F \sin [\nu(t)t + \phi(t)]\},$$

which gives:

$$1 - E_0 = \int_{E_0}^{1} dE = F \int_{0}^{t_{max}} u(t) \sin [\nu(t)t + \phi(t)]dt.$$

Under the resonant condition, the velocity $u(t)$ contains a term proportional to the driving and, hence, the integrand contains a term proportional to $\sin^2 [\nu(t)t + \phi(t)]$ which gives the major contribution that is proportional to $F t_{max}$ (the integration of the remaining terms are approximately zero because of the long-time average of the sinusoidal function). Since $t_{max} \sim n_{max}$, we have: $F n_{max} \sim \text{constant}$, which is the inverse relation (2.7).

After the energy reaches the threshold unity, the oscillatory behavior becomes chaotic. This is because the system has been pushed to moving near the hyperbolic points about which there is typically a homoclinic or heteroclinic tangle between the stable and unstable manifolds of the fixed points and consequently chaos [39, 40]. Because there is no dissipation, the system remains at $E = 1$ (and therefore remains chaotic) even if the external perturbation is withdrawn. Figure 5(a) shows the chaotic time series of $x$ (unwrapped with respect to $2\pi$). The phase-space trajectory $(x, y)$ on the stroboscopic section $t = 2\pi n/\nu(E = 1) \ (n = 1, \ldots)$ is shown in Fig. 5(b), where $0 \leq x \leq 2\pi$. The induced chaotic behavior near the hyperbolic fixed points can be seen more clearly in Fig. 5(c).
Figure 4. The relation between $n_{\text{max}}$, the number of oscillations of the external excitations required for energy $E$ to reach unity (i.e., for chaos to occur), and the perturbation strength $F$. 
Figure 5. For the nondissipative case, (a) induced chaotic time series $x_n$, (b) phase-space trajectory $(x, y)$ on the stroboscopic section defined by $t = 2\pi n/\nu(E = 1)$ ($n = 1, \ldots$), and (c) induced chaotic motion near the hyperbolic fixed point.
4.3.1. \textit{Determination of the largest Lyapunov exponent.} To confirm that chaos has indeed been induced, it is insightful to compute the largest Lyapunov exponent $\lambda_1$, as the positiveness of the exponent is the defining characteristic of chaos. A complication is that the system equations under external time-dependent perturbation are not totally known. As a result, the standard procedure for computing the Lyapunov exponents in deterministic systems [41] is not applicable. Neither is the system random, so that the methods for computing Lyapunov exponents in random dynamical systems, such as those based on products of random matrices [42, 43], are not suitable either. However, the largest exponent $\lambda_1$ can be estimated by simply monitoring how the length of a small phase-space vector changes with time. In particular, we start with a small vector $\Delta x(t_0)$ at $t_0 = 0$ and monitor its evolution up to time $t_1 > t_0$, where $t_1$ is the time at which the vector $\Delta x(t_1)$ is still small so that it can be related to $\Delta x(t_0)$ approximately linearly. The ratio $r_0 = |\Delta x(t_1)|/|\Delta x(t_0)|$ is then recorded and the vector $\Delta x(t_1)$ is normalized so that it is small in length. The new vector is evolved in time, and the same procedure yields a new ratio $r_1$, and so on. The largest exponent $\lambda_1$ is approximately given by:

$$\lambda_1 \approx \frac{1}{N} \sum_{i=0}^{N} \ln r_i, \text{ for } N \text{ large.}$$

Figure 6 shows $\lambda_1$ of system versus the perturbation strength $F$, where the positiveness of the exponent indicates induced chaotic motion.

The large fluctuations in the Lyapunov exponent in Fig. 6 are due to the “stickiness” effect of Kol’mogorov-Arnol’d-Moser (KAM) tori (surfaces) in Hamiltonian systems. Consider the general case where chaotic regions in the vicinities of hyperbolic fixed points coexist with some KAM surfaces. The stickiness effect is that, if a particle is initialized in the chaotic region, the particle wanders close to some KAM surface for a long time. Take, then, two nearby points on a given KAM surface and observe their evolution. What
one typically finds is that the distance between the two points changes little with time, because the Lyapunov exponents in the directions along the KAM surface are zero (i.e., the motion is quasiperiodic). The symplectic nature of Hamiltonian dynamics implies that the Lyapunov spectrum is organized in pairs of exponents with equal value but opposite signs. Hence, an orbit on a KAM surface has zero Lyapunov exponents in directions both along and perpendicular to the surface. Due to ergodicity, a particle initialized in the chaotic region will come arbitrarily close to some KAM surface. When this occurs, the effective Lyapunov exponents will be small, leading to slow divergence of the particle trajectory from the KAM surface. The result of the stickiness effect on the particle transport is that the particle-decay law becomes algebraic [44, 45, 46] in contrast to the exponential law observed in typical dissipative chaotic systems. Computationally, the consequence is an extreme slow convergence of dynamical invariants associated with chaotic orbits such as the Lyapunov exponents, which leads to large numerical fluctuations such as those observed in Fig. 6. As we will see later, when a small amount of dissipation is present, the fluctuations essentially disappear.

5. Nonlinear resonance in dissipative systems

We now turn to investigate the practically more relevant issue of dissipation. As we have seen, in the absence of dissipation, an arbitrarily small perturbation is enough to drive the system to the vicinity of the hyperbolic orbit. We can expect that dissipation will change this. The reason becomes clear if we examine our method from the point of view of energy transfer. What the method does is to ensure that energy is always transferred from the external perturbation to the system. That is, we make sure that the external forcing is always injecting energy into the system (and never extracting from it). With no dissipation,
Figure 6. Estimated largest Lyapunov exponent $\lambda_1$ versus the perturbation strength $F$. The positiveness of the exponent indicates that chaos has been induced.
Figure 7. For $F = 10^{-4}$, (a) Energy of the oscillator as a function of time for an excitation at constant frequency for the dissipative system, where $\alpha = 5 \times 10^{-5}$. (c) Frequency of the applied perturbation as a function of time for $\alpha = 5 \times 10^{-5}$. (b,d) Same as in (a,c) respectively, except that the dissipation parameter is $\alpha = 7.5 \times 10^{-5}$. 
Figure 8. For $F = 10^{-4}$: (a) the maximally reachable energy $E_{max}$ of the system versus the dissipation parameter $\alpha$, (b) the time $n_{max}$ (in number of cycles of the external perturbation) required to reach $E_{max}$ versus $\alpha$. 
the system’s energy just keeps increasing (albeit slowly), until the system inevitably reaches the energy of the hyperbolic orbit. This picture changes in the presence of dissipation: although we still inject energy into the system using the external perturbation, now energy is also getting lost because of the dissipation. For a given energy, call the average energy input rate due to the forcing $\kappa_{in}$ and the energy output rate due to dissipation $\kappa_{out}$. Then, $\kappa_{out}$ usually increases the farther the system is from the equilibrium point, which could be a stable fixed point or a stable cycle, for example. The total energy of the system will stop increasing when $\kappa_{out}$ equals $\kappa_{in}$. There are two possible scenarios: (1) if this happens for an energy above the energy of the hyperbolic orbit $E_{max}$, then we will be able to achieve the goal of exciting the system to near $E_{max}$ and therefore inducing chaos, and (2) if, however, $\kappa_{out}$ becomes equal to $\kappa_{in}$ for an energy $E$ less than $E_{max}$, then the system will saturate at that energy, and we will not be able to push it to the neighborhood of the hyperbolic orbit. We can expect that, for a fixed forcing amplitude, as the dissipation increases from 0, a transition from case (1) to case (2) will occur.

These ideas can be tested through a numerical experiment. We use Eq. (2.6) with an added dissipative term proportional to the velocity:

$$\frac{d^2 x}{dt^2} = -\frac{dV(x)}{dx} - \alpha \ddot{x} + F \sin [\nu(t)t + \phi(t)],$$

(2.8)

where $\alpha$ is the dissipation coefficient. The frequency and phase functions $\nu(t)$ and $\phi(t)$ of the perturbation are determined using our method, in exactly the same way as in the nondissipative case. For small enough dissipation, the system’s oscillations are again nearly periodic, and all the assumptions we used for the calculation of $\phi$ and $\nu$ as described in Sec. 4 remain valid. The two cases where chaotic motion can and cannot be excited can be seen by fixing the perturbation strength $F$ and initial condition (the same as that used for the nondissipative case) and plotting the energy as a function of oscillation period for
two values of the dissipation $\alpha$. The result is displayed in Fig. 7, where Fig. 7(a) denotes the case where the energy can be increased to unity for $\alpha = 5 \times 10^{-5}$, and Fig. 7(b) is the over damping case where the energy cannot be increased to unity for $\alpha = 7.5 \times 10^{-5}$. Figures 7(c) and 7(d) show the frequency variations corresponding to Figs. 7(a) and 7(b), respectively.

One interesting feature in dissipative systems is that, when chaos is induced, in order to maintain it, the frequency of the external excitation needs to be adjusted continuously to keep the energy of the system at about 1, as shown in Fig. 7(c). The reason is that, when the system is driven to chaos, external energy is still needed to be delivered to the system to keep it in the chaotic state due to the dissipation. That is, it is still necessary to make the external perturbations resonant with the system. However, if continuous resonant perturbations are applied, it is likely that the system will settle into a faster and faster rotational motion with monotonically increasing energy. Such a motion is in fact not chaotic. To avoid this situation, we monitor the dynamical variable $x(t)$ (in practice, this can be measured). If it exceeds $2\pi$, we turn off the time variations in $\nu(t)$ and $\phi(t)$ so that the resonant condition is temporally not satisfied. As a result, little energy is transferred into the system so that its energy starts to decrease, at which point we turn on the time variations of $\nu(t)$ and $\phi(t)$ so that the condition of resonance is fulfilled again, and so on. This results in a continuous but small change in the frequency $\nu(t)$, as indicated in Fig. 7(c). In contrast, in the nondissipative case, after chaos is induced, the system energy can be maintained at about $E = 1.0$ even when the time variations in $\nu(t)$ and $\phi(t)$ are turned off [Fig. 3]. In other words, if there is no dissipation, after chaos is induced the external perturbations need not to be resonant with the system to maintain the induced chaotic motion.

For a given value of the perturbation strength $F$, there exists a maximum amount
Figure 9. For $F = 10^{-4}$ and $\alpha = 5 \times 10^{-5}$, (a) chaotic time series $x_n$, (b) phase-space trajectory $(x_n, y_n)$ on a stroboscopic section, and (c) blowup of the trajectory near the hyperbolic fixed point where chaos is apparent.
Figure 10. For $F = 10^{-4}$, the largest Lyapunov exponent $\lambda_1$ of the perturbed dynamics versus $\alpha$, where the exponent is clearly positive for $\alpha < \alpha_c$. 
Figure 11. For fixing $\alpha = 5 \times 10^{-5} < \alpha_c$, the time $n_{\text{max}}$ required to achieve chaos versus the perturbation strength $F$. The scaling between $n_{\text{max}}$ and $F$ is similar to that in the nondissipative case.
of the dissipation beyond which the system cannot be driven to chaos. Figures 8(a) and 8(b) show, for \( F = 10^{-4} \), the maximally reachable energy of the system and the time (in number of cycles) required to reach \( E_{\text{max}} \) versus the dissipation parameter \( \alpha \), respectively. We see that for \( \alpha < \alpha_{\text{cr}} \approx 6 \times 10^{-5} \), the system can be driven to the hyperbolic point with \( E_{\text{max}} = 1 \). The time it takes for the system to become chaotic is in general longer than that required in the nondissipative case. For \( \alpha > \alpha_{\text{cr}} \), however, the energy never reaches \( E_{\text{max}} \), and it saturates at an energy that becomes smaller as \( \alpha \) increases, as expected. The critical dissipation \( \alpha_{\text{cr}} \) defines the threshold that separates the two cases where the system can be excited to the hyperbolic energy and cannot be, in the latter the dissipation is strong enough to prevent the system from reaching chaos. We observe that the value of the critical dissipation \( \alpha_{\text{cr}} \) increases approximately linearly with \( F \), indicating (expectedly) that the external excitation needs to be stronger to induce chaos for systems with greater dissipation. Although we use a particular system for numerical simulations, it is clear that these features are quite general.

To confirm that the system is indeed chaotic when \( E_{\text{max}} = 1.0 \) is reached for \( \alpha < \alpha_c \), we plot in Figs. 9(a-c) the time series \( x_n \), a phase-space trajectory, and a blowup of the behavior of the trajectory near the hyperbolic fixed point, respectively. These plots, in particular Fig. 9(c), point to a chaotic behavior. Figure 10(a) shows, for \( F = 10^{-4} \), the largest Lyapunov exponent versus \( \alpha \), where we see that \( \lambda_1 > 0 \) for \( \alpha < \alpha_c \), confirming that the dynamics of the system under external perturbation is indeed chaotic. Also note that, because of dissipation, the convergence of the computed Lyapunov exponent is good (compared with Fig. 6). We find that, for fixed dissipation parameter \( \alpha < \alpha_c \), the time \( n_{\text{max}} \) required to achieve chaos is also inversely proportional to the perturbation strength \( F \), as shown in Fig. 11.
CHAPTER 3

Experimental Verification

"Argument is conclusive... but... it does not remove doubt, so that the mind may rest in the sure knowledge of the truth, unless it finds it by the method of experiment. For if any man who never saw fire proved by satisfactory arguments that fire burns, his hearer’s mind would never be satisfied, nor would he avoid the fire until he put his hand in it that he might learn by experiment what argument taught."

- Roger Bacon

1. The Duffing Oscillator

We base our experimental verification by using a Duffing oscillator as the target non-linear system which will be driven into chaos from a stable periodic state. The Duffing oscillator is chosen since it can produce a wide variety of robust periodic and chaotic outputs. To produce a signal that is matched to the natural frequency of the oscillator, we use a Phase-Locked Loop that is designed to work in the operating range of our Duffing oscillator.

1.1. Description.
Consider a general driven system given by
\[ \frac{d^2x}{dt^2} = f(t) \] (3.1)
which describes the oscillation of a system driven by a force \( f(t) \). If there is a damping force present in the system, the system can be described as
\[ \frac{d^2x}{dt^2} + \delta \frac{dx}{dt} = f(t) \] (3.2)
where the damping force is proportional to the velocity \( \frac{dx}{dt} \). This is a second order system which is still linear. Now if we introduce a non-linear term \( x - x^3 \), we end up with the following system which is known as the Duffing oscillator.
\[ \frac{dx}{dt^2} + \delta \frac{dx}{dt} - x + x^3 = f(t) \] (3.3)
The system described by Eqn. 3.2 is easy to realize, but the non-linear term \( -x + x^3 \) in Eqn. 3.3 makes it difficult to build a circuit that follows the equation exactly. This difficulty is overcome by breaking down \( -x + x^3 \) into piecewise linear regions which can be easily implemented in a circuit.

1.2. Piecewise Linear Approximation. Let,
\[ w(x) = -x + x^3 \] (3.4)
To form the piecewise linear model, we compute the slope of the approximate line. Differentiating (3.4), we get
\[ w'(x) = -1 + 3x^2 \] (3.5)
Since \( w(0) = 0, w'(0) = -1 \) and \( w(\pm 1) = 0, w'(\pm 1) = 2 \), we can represent \( w(x) \) by three lines i.e., a line of slope -1 passing through \((0,0)\), and lines of slope 2 passing through \((\pm 1,0)\). The piecewise linear form of \( w(x) \) can be written as
Figure 12. Piecewise linear approximation for $x - x^3$. 
Figure 13. RLC circuit that forms the basis of the Young-Silva oscillator. The signal $V_0$ is designed according to (3.11).

\[
w(x) = \begin{cases} 
-x & \text{inside}, \\
2x + k & \text{outside}. 
\end{cases} \quad (3.6)
\]

This piecewise linear approximation along with the inner and outer regions is shown in Fig. 12 and it forms the basis of the Young-Silva Chaotic Oscillator.

1.3. The Young-Silva circuit implementation.

Consider a simple RLC circuit as shown in Fig. 13. Solving the circuit equations, we get

\[
V_0 = L\ddot{X}_2 + R_1X_2 + X_1 \quad (3.7)
\]
\[
X_2 = C\dot{X}_1 + \frac{X_1}{R_2} \quad (3.8)
\]
\[
V_0 = LC\ddot{X}_1 + \left(\frac{L}{R_2} + R_1C\right)\dot{X}_1 + \left(1 + \frac{R_1}{R_2}\right)X_1 \quad (3.9)
\]
Combining the above equations and considering $R_1 << R_2$, we get

$$V_0 = LC \ddot{X}_1 + \left( \frac{L}{R_2} + R_1 C \right) \dot{X}_1 + X_1$$  \hspace{1cm} (3.10)

Comparing (3.10) with (3.3), and using the piecewise linear form of $w(x)$ in (3.6), we can write

$$V_0 = \begin{cases} 
  f(t) + 2X_1 & \text{inside}, \\
  f(t) - X_1 & \text{outside}. 
\end{cases} \hspace{1cm} (3.11)$$

Thus, in order to realize the duffing oscillator, we need only use the simple RLC circuit in Fig. 13, where $V_0$ is designed as above. This can be seen in Fig. 14 where an op-amp difference amplifier and a diode limiter are used to generate $V_0$ as a sum of the driving function $f(t)$ and either $2X_1$ or $-X_1$. The rest of the circuit consists of op-amp voltage followers, used after the input and before outputs as buffer amplifiers. The complete circuit is shown in Fig. 15.
Figure 15. Complete circuit for the Duffing oscillator. $X_1$ is the driving signal and the outputs are measured from $Y_1$ and $Y_2 - Y_3$. Op-amps are used to generate $Y_2 - Y_3$ and to buffer the inputs and outputs for measurement.
Figure 16. Block diagram of a phase-locked loop. The output $V_2$ of the VCO (voltage controlled oscillator) is matched in frequency and synchronized in phase with the input $V_1$ to the phase detector.

2. The Phase Locked Loop

A phase-locked loop is a circuit that synchronizes a signal with another signal in both frequency and phase. Our phase-locked loop is used to generate a signal $V_2(t)$ that has the same frequency, and is synchronized in phase with an input signal $V_1(t)$. The input signal is from the Duffing circuit with a periodic attractor, and the output signal is the resonant perturbation to be applied to the Duffing circuit. We expect this match in frequency and phase of oscillations of the target nonlinear system (the Duffing circuit) with the frequency and phase of the input signal to drive it into a hierarchy of higher resonant states, and ultimately into chaos. It should be noted that this is the essential requirement for driving the target system to chaos in our method. The phase-locked loop also introduces a finite delay in the feedback loop which gives the system time to respond to the change in frequency and phase of the perturbation.

2.1. Components of the PLL. The basic block diagram of a phase-locked loop is shown in Fig 16, which consists of a phase detector (PD), a voltage controlled oscillator (VCO), and a loop filter (LF). The phase detector generates an output voltage $V_{pd}(t)$ proportional to the phase error $\theta_e$ between the input signal $V_1(t)$ and the output of the VCO.
Figure 17. Schematic diagram of the phase-locked loop circuit implemented using an LM565 chip in our experiments. The center frequency of the VCO is adjusted using the variable resistor $R_1$. 
\( V_2(t) \), as follows:

\[
V_{pd}(t) = K_{pd} \theta_e, \tag{3.12}
\]

where \( K_{pd} \) is the PD gain. The alternating current components of the output of the PD are filtered out by the loop filter. The output of the VCO has a frequency that depends on the voltage \( V_f(t) \) input to it and is given by

\[
\omega_2(t) = \omega_0 + K_v V_f(t) \tag{3.13}
\]

where \( \omega_0 \) is the center frequency of the VCO and \( K_v \) is the VCO gain.

2.2. Working mechanism. The PLL works by matching the output frequency \( \omega_2 \) of the VCO with the input frequency \( \omega_1 \). The phase detector compares the two input frequencies and produces a signal proportional to the phase difference between them. If the frequencies \( \omega_1 \) and \( \omega_2 \) do not match, the phase detector produces a signal (called the phase-error signal) which causes the output frequency \( \omega_2 \) of the VCO to move in the direction of the input frequency \( \omega_1 \).

If the input \( V_1(t) \) is absent, the VCO oscillates at a frequency \( \omega_0 \) known as the center frequency or free running frequency of the phase-locked loop. The center frequency can be easily adjusted to any desired value by small design changes, and can enable us to operate the PLL at a suitable initial frequency. The capture range \( \Delta \omega_C \) of a phase-locked loop is the range of frequencies over which it can initially acquire a lock. Once a lock has been acquired, the phase-locked loop will continue to track the input frequency over a range known as the lock range \( \Delta \omega_L \). If the input frequency varies beyond the lock range, the locked loop will become unlocked. The capture range is never greater than the lock range.
Since our Duffing oscillator is designed to operate in the 1-15kHz range the PLL is designed such that the capture and lock ranges are within the frequency range of interest.

An important component of the phase-locked loop is the loop filter $G(s)$ which filters out noise present in the input signal and prevents the circuit from locking on to unwanted harmonics in the input. This feature makes it possible to track the Duffing frequency and drive it using resonant perturbations even in the presence of noise.

2.3. Circuit setup. In our experiments, an LM565 phase locked loop integrated circuit is used whose internal circuitry is shown in Fig. 17. The center frequency is given by

$$\omega_0 = \frac{0.6\pi}{R_1C_1},$$

and the lock range is

$$\Delta \omega_L = \pm \frac{8\omega_0}{V_{cc}}$$

where $V_{cc}$ is the supply voltage to the circuit. The output of the phase-locked loop at pins 4 and 5 is a square wave with the same frequency and phase as the input signal at pin 2. We thus obtain a square-wave that is frequency and phase matched with the output of the Duffing oscillator.

3. Complete experimental scheme

The complete experimental scheme is summarized in the block diagram in Fig. 18. The output $Y_1$ of the Duffing oscillator is fed to one of the inputs of the phase detector and the output of the VCO is used to feed the resonant perturbation to the Duffing oscillator.
Figure 18. Block diagram of the experimental scheme: the phase-locked loop is used to track the output frequency of the oscillator, and a signal at this frequency is used to drive the oscillator with the delay introduced by the phase-locked loop. A voltage divider circuit with a potentiometer is used to vary the output amplitude. The outputs are fed to a DAQ system for measurement and analysis.

A voltage divider at the output of the VCO is used to adjust the amplitude of the driving signal. \( R_p \) was taken to be a 10KΩ potentiometer and \( R_c \), a 1KΩ resistor. The phase-locked loop produces a signal that is matched in frequency and phase with the intrinsic oscillations of the Duffing oscillator, which is then fed back at an appropriate amplitude to the Duffing circuit. As will be seen later, the potential divider is necessary since the response of the system to the resonant signal depends on the amplitude of perturbation.

3.1. The data acquisition system. Data acquisition (DAQ) is an important component of the experimental setup since the experimental process generates large amounts of data which has to be collected and processed efficiently. Our setup uses a National Instruments PXI Chassis which is a complete measurement and automation system with modules for various applications and a built-in computer for storing and processing data. An analog data acquisition module is used for our purpose since it has the capability to
acquire multiple channels of data simultaneously with a buffer size that is limited only by the size of the computer memory.

The acquisition process is controlled by a LabView Virtual Instrument (VI) which is written specifically for the circuit setup used. The Duffing circuit has two measurable outputs Y1 and Y2 as shown in Fig. 15. These outputs are buffered using op-amps to provide adequate protection to the measurement system and to prevent any loading of the Duffing circuit due to the measurement process. The buffer in the DAQ system is configured to collect data in a single shot until the specified number of data points are collected at the desired sampling rate. Data is then unloaded from the buffer in blocks which are then
processed and written to MATLAB readable files. This process ensures that there are no ‘breaks’ in data or overwriting of data in the buffer due to incoming data which would happen if data was acquired continuously. A rough estimation of the frequency of oscillation of the Duffing oscillator shows that it is very close to the driving frequency in the frequency region of interest (1-15kHz). It is therefore possible to easily choose a sampling frequency and data length which is adequate to represent the attractors formed. A screenshot of the acquisition process using LabView is shown in Fig. 19.

4. Observations

The Duffing oscillator can give a variety of outputs depending on the frequency of the driving signal. Our goal is to apply resonant perturbations to the oscillator when it is operating naturally in a periodic state and drive it into a chaotic state. The observations from our experiment are summarized in the following sections.

4.1. Duffing oscillator operation. The operation of a Duffing oscillator is summarized in Fig. 20 which shows a bifurcation plot. A bifurcation plot shows the output of the system as a function of a parameter. Points on the plot are obtained by taking a Poincaré surface of section which is a method of reducing a $N$-dimensional continuous time system (flow) to a $(N - 1)$-dimensional map as shown in Fig. 21. The plot can be understood in the following manner: For each frequency shown in the $x$-direction, the number of points in the $y$-direction gives the periodicity of the output. Thus two points signify a period-2 orbit while 4 points indicate a period-4 orbit. If there are a large number of points for a particular frequency, it indicates that the output is probably chaotic which can be verified
Figure 20. Bifurcation plot for the Duffing oscillator driven by a square wave. As the driving frequency is varied from 1-7.4kHz, a wide range of periodic and chaotic orbits can be seen.
Figure 21. Generation of a Poincaré map from a flow. The points on the map are obtained as piercings on the Poincaré surface of section.

visually from the corresponding phase-space plot. Fig. 20 also shows the phase-space plots for some of the frequencies.

It can be seen that the Duffing oscillator has large regions where the output is periodic and small regions where the output is chaotic. There are also small periodic windows within the chaotic regions. In our experiment, we will operate the oscillator in a periodic region which is far away from any chaos and attempt to take it to a state that is chaotic. The nature of the Duffing output also depends on the amplitude of the driving signal. Thus a driving signal which produces a periodic output may produce a chaotic output at the same frequency but a different amplitude. It will be verified later that chaos is induced due to the perturbations being resonant, and not due to the amplitude of the perturbations.
Figure 22. (a) Phase-space plot of the output of the Duffing circuit in stable state with horizontal and vertical axis in volts (b) Time domain plot of Y1 with vertical axis in volts (c) Prediction error (d) Distribution of the maximal Lyapunov exponent
4.2. System injected with resonant perturbations. This setup retains the driving signal (sinusoidal or square wave) for the Duffing oscillator and injects the resonant perturbation in the circuit at the point $X$ as marked in Fig. 15. Here the dimensionality of the system is increased due to the resonant perturbation. Fig. 22a,b show the phase and time-domain plots when the amplitude of the square-wave driving signal is kept at $4V_{p-p}$ and the frequency is set at 4.5kHz which results in a periodic output from the Duffing oscillator. Fig. 22c shows the prediction error computed as described in Sec. 3.2.2. It can be seen that the prediction error does not increase with time which is an indication of a stable orbit. The histogram of the maximum Lyapunov exponent is shown in Fig. 22d and it can be seen that $\lambda_1 \approx 0$, confirming the periodic nature of the oscillations.

Next, the PLL is made to track the Duffing output and is connected to the point $X$ through a $1K\Omega$ resistor with an amplitude of $\approx 1.32V_{p-p}$. The amplitude of the driving signal is not changed while switching from a plain driving to a resonant driving. Fig. 23a,b indicate apparent chaos in the Duffing oscillator. To verify that the oscillator is indeed in chaos, we calculate the prediction error (Fig. 23c) which increases monotonically. The histogram of the maximal Lyapunov exponent is shown in Fig. 23d which shows that $\lambda_1 \approx 1.3 \times 10^{-3}$. As explained in Sec. 3.2.2 a positive Lyapunov exponent is an indication of chaos.

4.3. System injected with noise. The above system setup gives rise to the question of whether the transition to chaos is due to the injected signal being resonant or merely due to a disruption in the system dynamics due to the addition of a new signal. This issue will be addressed by adding noise instead of a resonant signal. Even at very high noise levels ($6V$ through a $1K\Omega$ resistor) the system output is seen to merely smear and become
Figure 23. (a) Phase-space plot of the output of the Duffing circuit under resonant driving with horizontal and vertical axis in volts (b) Time domain plot of Y1 with vertical axis in volts (c) Prediction error (d) Distribution of the maximal Lyapunov exponent
Figure 24. (a) Phase-space plot of the output of the Duffing circuit under noisy driving with horizontal and vertical axis in volts (b) Time domain plot of Y1 with vertical axis in volts (c) Prediction error (d) Distribution of the maximal Lyapunov exponent
noisy instead of transitioning to a chaotic state as seen in Fig. 24a,b. Further proof is provided in Fig. 24c which shows that the prediction error does not increase and is similar to the prediction error in Fig. 22 for the periodic attractor. The histogram of the maximal Lyapunov exponent in Fig. 24 shows that $\lambda_1 \approx 0$ which indicates that the output is periodic and that the smear in the phase plot is merely due to noise and not chaos. Our initial system condition was intentionally chosen to be very far from a chaotic regime and it can be inferred from this experiment that the simple addition of noise is incapable of taking the system from the periodic state to a chaotic one.

4.4. Performance of the circuit in noisy conditions. As mentioned in Sec 2 the loop filter present in the phase-locked loop helps filter out noise and prevents it from locking on to unwanted harmonics in the input. This was verified by using the same setup as described in Sec 4.2 while adding noise through a difference amplifier to the input. This makes the input periodic driving noisy and creates a noisy output. It was verified that on applying resonant perturbations the system went into chaos as before up to a noise amplitude of $2 V_{p-p}$ which provides sufficient indication that the system works well even in the presence of considerable ambient noise.
CHAPTER 4

Conclusions and Future Work

This thesis explored the idea of inducing chaos in systems using resonant perturbations. An overview of the existing work related to the topic of inducing chaos is presented. The mathematical model of inducing chaos using resonant perturbations is described in detail with reference to both dissipative and non dissipative systems. Finally the model is put to test using an experimental setup which uses a Duffing oscillator as the nonlinear system and a phase locked loop as the mechanism to generate resonant perturbations.

Resonant perturbations are seen to be capable of inducing chaos in the Duffing oscillator normally operating in a periodic state. It is verified that this transition to chaos is not possible by using random perturbations. This verification is done by replacing the resonant perturbations with noise of varying amplitudes.

It is observed that while resonant perturbations are capable of inducing chaos in a previously nonchaotic system, the amplitude of perturbations required to induce chaos varies with system parameters. However, in general it is observed that for any value of system parameters, there is some amplitude or range of amplitudes of the resonant perturbations which is capable of taking the system to a chaotic regime. The experimental setup used is largely immune to external noise since the loop filter in the phase locked loop ensures that it does not track noisy signals or harmonics in the input.
It is interesting to generate a general algorithm to induce chaos in any given system simply by observing the outputs of the system. For this it would be necessary to implement a setup that can sense the chaotic nature of the output in real time and make adjustments to the amplitude of perturbations until chaos occurs.

Another topic which needs to be explored to obtain a better understanding of the mechanism of inducing chaos is the transition to chaos. Systems may reach a chaotic regime from a periodic regime in a variety of ways including period-doubling bifurcations, intermittency and crises. The exact nature of the route to chaos would provide a better insight into creating mechanisms to induce chaos for any given system.
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